

Solution to the Static Charge Distribution on a Thin Wire Using the Method of Moments

James R. Nagel

Department of Electrical and Computer Engineering

University of Utah, Salt Lake City, Utah

April 2, 2012

1 Introduction

Just as differential equations are a common class of mathematical tools for electromagnetic systems, integral equations can likewise provide valuable insight into practical engineering problems. By analogy, finite-difference methods serve as a popular tool for numerically solving differential equations while the method of moments (MoM) is one of the more popular tools for solving integral equations. This makes MoM a cornerstone for any comprehensive study in numerical methods. Unfortunately, the essence of MoM is much less intuitive than traditional finite-differencing methods. To make matters even worse, effective tutorials are very scarce throughout the literature. The reason for this is because a full discussion of MoM generally requires a firm grasp of linear vector spaces and variational calculus. In fact, some standard references will happily devote entire chapters to these topics alone before even attempting to introduce MoM [1].

The goal of this paper is to introduce the core concepts of MoM by applying it to a relatively simple physical problem: the charge distribution along a thin wire of constant potential. This problem allows us to walk through the principles of MoM while still maintaining a simple, intuitive framework based on physical principles. The problem was originally explored as an undergraduate tutorial in [2], though some mathematical errors and misconceptions prevent it from being as instructive as it was intended. We shall therefore explore the problem again from a more fundamental perspective so that introductory-level students may grasp the core of MoM without having to spend time delving into more advanced mathematical concepts.

2 Voltage Potential Due to an Arbitrary Charge Distribution

We begin our discussion with a point charge q placed at the origin of an arbitrary coordinate system. From Coulomb's law, we know that the electric field intensity at some point \mathbf{r} away from the origin satisfies

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}, \quad (1)$$

where ϵ_0 is the permittivity of free space. We also note the use of standard vector notation defined by

$$\mathbf{r} = x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}, \quad (2)$$

$$r = |\mathbf{r}|, \quad (3)$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}. \quad (4)$$

Now let us solve for the voltage potential V at the same point \mathbf{r} due to the presence of the point charge. Because we are dealing with an electrostatic system, we know that the electric field

is simply the negative gradient of the voltage potential field:

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -\nabla V(\mathbf{r}) \\ &= -\hat{\mathbf{x}} \frac{\partial V}{\partial x} - \hat{\mathbf{y}} \frac{\partial V}{\partial y} - \hat{\mathbf{z}} \frac{\partial V}{\partial z} .\end{aligned}\tag{5}$$

Due to the radial symmetry in this problem, we are free to solve for V at any convenient observation point we desire and then simply apply the result to all other points along a sphere with equivalent radius. Let us therefore choose $\mathbf{r} = x \hat{\mathbf{x}}$ to represent some arbitrary point along the x -axis. Solving for \mathbf{E} then yields

$$\mathbf{E}(x) = \frac{1}{4\pi\epsilon_0} \frac{q}{x^2} \hat{\mathbf{x}} .\tag{6}$$

Comparing this expression with $-\nabla V$ then leads us to

$$\frac{\partial V}{\partial x} = -\frac{1}{4\pi\epsilon_0} \frac{q}{x^2} .\tag{7}$$

Next, we integrate with respect to x to find

$$V(x) = \frac{q}{4\pi\epsilon_0 x} + C .\tag{8}$$

Note that we still need to define C in order to arrive at a unique solution for V . One convenient way to handle this is by arbitrarily specifying the voltage potential at $x = \infty$ to be zero. This forces a value of $C = 0$ and uniquely defines V to be

$$V(x) = \frac{q}{4\pi\epsilon_0 x} .\tag{9}$$

Finally, because of the radial symmetry of the system, whatever is true at the point x must also be true for all points along the surface of a sphere with radius $r = x$. Expressing V in spherical coordinates therefore produces a modification of Coulomb's law in terms of the voltage potential function:

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r} .\tag{10}$$

In practice, it is usually not possible for our charges to lie exactly at the origin of the system. A far more useful form of Coulomb's law therefore accounts for charges that may be offset to some arbitrary point \mathbf{r}' . In this case, the distance between the source and the observation point is written as $r = |\mathbf{r} - \mathbf{r}'|$. Thus, a more generic form of Coulomb's law may be written as

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}'|} .\tag{11}$$

Next, we desire to account for the presence of multiple point charges in space. Fortunately, this is an easy problem to handle since the electric fields add linearly. This means that the total electric field due to a system of N point charges is simply the summation of all the individual fields. Consequently, the total voltage potential due to a system of N point charges is also the summation of all the individual potentials. Letting q_n and \mathbf{r}'_n denote the charge and position of the n th source therefore leads us to

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \frac{q_n}{|\mathbf{r} - \mathbf{r}'_n|} .\tag{12}$$

Finally, we are now in a position to describe the total voltage potential due to a distribution of charge density ρ rather than a discrete series of point charges. This situation is handled by simply replacing the finite summation over points charges with an integral over a volume of charge density. The individual point charges in q_i are then replaced with the differential charge $dq = \rho(\mathbf{r}')dV'$, where ρ is the charge density function and dV' is the differential unit of volume. Integrating over the entire distribution of charge within the source volume V' then leads us to

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'} , \quad (13)$$

which is Coulomb's law for voltage potential in its most generic form.

3 Forward Versus Inverse Problems

Upon inspection of Equation (13), it is tempting to physically interpret it in the following manner:

Given an arbitrary distribution of charge density ρ , find the voltage potential V throughout all space.

Such an interpretation is intuitive and natural for many practical applications, so we generally refer to it as the *forward problem*. It should also be straightforward to see how numerical integration allows us to obtain solutions for virtually any physical geometry imaginable. However, there is an alternative way to view Equation (13):

Given the voltage potential V throughout all space, find the charge density ρ that produced it.

This interpretation is called the *inverse problem*. Though it may seem less intuitive than the forward problem, it is still mathematically possible to perform under many physical conditions. The only hard part is trying to find a way to invert the integration operation and solve for ρ . As we shall see, exact solutions to the problem can be difficult (if not impossible) to obtain, but numerical methods are available to help us generate approximate solutions with a high degree of accuracy. One of the most popular of these is the *method of weighted residuals*, or more generally the *method of moments* (MoM).

4 Charge Distribution on a Wire of Constant Potential

To understand the essence of how MoM works, it is instructive to demonstrate on a simplified model. One very classic example is the system illustrated in Figure 1, which shows a thin metal rod of length L and radius a held at constant potential V_0 . Because the rod is held at some voltage potential relative to its environment, there naturally exists some distribution of charge density ρ that spreads out over the surface. Our goal is to therefore solve for ρ given V_0 , L , and a .

The solution begins by rewriting Equation (13) in one dimension. As part of the "thin rod" assumption ($a \ll L$), we can approximate the charge distribution as if it were placed entirely along the x -axis. We shall furthermore limit our observation point \mathbf{r} strictly to points along the x -axis since doing so will help simplify the notation without changing the essence of the problem. The voltage potential at some point x may therefore be expressed as

$$V(x) = \frac{1}{4\pi\epsilon_0} \int_0^L \frac{\rho(x')}{|x - x'|} dx' . \quad (14)$$

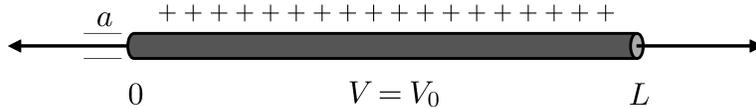


Figure 1: A thin, conductive rod with length L and radius a is oriented along the x -axis. The rod is fixed at a voltage potential of V_0 , thus inducing a charge density $\rho(x)$ along its surface.

Equations with this form have a special name in mathematics, and they are called *Fredholm integral equations*. More specifically, Equation (14) is a *Fredholm equation of the first kind*. In general, such equations are written as

$$g(t) = \int_a^b K(x, t)\Phi(x) dx . \quad (15)$$

The function $K(x, t)$ is called the *kernal* while $g(t)$ is called the *forcing* function. Since Φ is the function we are trying to solve for, we may simply refer to it as the *unknown* function. We also note that because Equation (14) is similar in nature to Equation (15), whatever method that solves our charged rod problem can in general be applied to any number of problems that are likewise expressed as a Fredholm equation. This is what makes the method of moments such a powerful tool in numerical methods.

5 Basis Functions

The first step in applying the method of moments is to slice up the metal rod into a series of discrete subsections, each with a width Δx . This arrangement is shown in Figure 2 with the center of each segment located at the point x'_n . We then define an estimation function $\hat{\rho}$ that approximates the unknown function ρ by expressing it as a linear combination of discrete basis functions. Letting u_n denote the basis functions and α_n denote the weighting coefficients, this is written as

$$\hat{\rho}(x') = \sum_{n=1}^N \alpha_n u_n(x') . \quad (16)$$

The exact choice for u_n can vary, but one of the simplest choices is the Dirac delta function:

$$u_n(x') = \delta(x' - x'_n) . \quad (17)$$

This choice attempts to approximate the complete charge distribution on the rod as if it were a series of point charges located at the points x'_n . The exact values for x'_n are likewise arbitrary, but a natural choice is at the center of each subdomain such that $x'_n = n\Delta x/2$. Another common choice is the *rectangle* function, or simply *rect* function, defined by

$$u_n(x') = \begin{cases} 1, & |x' - x'_n| \leq \Delta x/2 \\ 0, & \text{otherwise} . \end{cases} \quad (18)$$

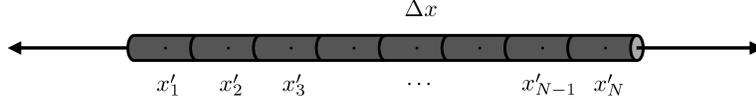


Figure 2: The charged rod is segmented into a series of subsections with length Δx .

Clearly, this approximates the charge distribution as a series of constant charge density values over each segment. Many other options exist as well, but for the sake of this example we shall continue on by applying delta functions as the basis for $\hat{\rho}$.

6 Residuals

Now that we have settled on a choice of expansion function, our next goal is to solve for the set of expansion coefficients α_n that best approximate the true solution ρ . We therefore begin by substituting $\hat{\rho}$ for ρ into Equation (14) to find

$$\begin{aligned}
 V(x) &\approx \frac{1}{4\pi\epsilon_0} \int_0^L \frac{1}{|x-x'|} \sum_{n=1}^N \alpha_n \delta(x'-x'_n) dx' \\
 &\approx \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \alpha_n \int_0^L \frac{\delta(x'-x'_n)}{|x-x'|} dx' \\
 &\approx \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \frac{\alpha_n}{|x-x'_n|}.
 \end{aligned} \tag{19}$$

The next step is to calculate the *residual* by subtracting the right side of Equation (19) from the left:

$$R(x) = V(x) - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \frac{\alpha_n}{|x-x'_n|}. \tag{20}$$

Note how the residual simply represents the error in our integral equation that arose after replacing ρ with an approximation function $\hat{\rho}$. Thus, a natural goal is to solve for a specific set of expansion coefficients α_n that *minimize the residual*. Unfortunately, this is not yet mathematically possible in its current form because R is a continuous function over x and there are still N unknown expansion coefficients in the summation.

To get around this problem, consider the voltage at the position $x = L/2$. Because the charges are distributed on a conductive metal, the entire wire is at the same voltage potential V_0 . The residual at this point therefore satisfies

$$R(L/2) = V_0 - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \frac{\alpha_n}{|L/2-x'_n|}. \tag{21}$$

In principle, we may not be able to drive R to zero over all space, but it is certainly possible to choose an appropriate set of α_n values such that $R(L/2) = 0$. In fact, because there are a total of N expansion coefficients, it is possible to drive at least this many samples in R to zero. To see how, suppose we were to sample a random set of M voltages along the wire such that $V(x_m) = V_0$. If we force $R(x_m) = 0$ at each location, the result is a system of equations given by

$$\begin{aligned} V_0 - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \frac{\alpha_n}{|x_1 - x'_n|} &= 0 \\ V_0 - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \frac{\alpha_n}{|x_2 - x'_n|} &= 0 \\ V_0 - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \frac{\alpha_n}{|x_3 - x'_n|} &= 0 \\ &\vdots \\ V_0 - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \frac{\alpha_n}{|x_M - x'_n|} &= 0. \end{aligned}$$

Note how the above expression represents a system of M linear equations with N unknowns. If we therefore specify $M = N$, we are guaranteed to find a unique solution for all values of the expansion coefficients in α_n . We may therefore express the above system as a matrix-vector equation with the form $\mathbf{Ax} = \mathbf{b}$. Writing this out explicitly results in

$$\begin{bmatrix} \frac{1}{|x_1-x'_1|} & \frac{1}{|x_1-x'_2|} & \frac{1}{|x_1-x'_3|} & \cdots & \frac{1}{|x_1-x'_{N-1}|} & \frac{1}{|x_1-x'_N|} \\ \frac{1}{|x_2-x'_1|} & \frac{1}{|x_2-x'_2|} & \frac{1}{|x_2-x'_3|} & \cdots & \frac{1}{|x_2-x'_{N-1}|} & \frac{1}{|x_2-x'_N|} \\ \frac{1}{|x_3-x'_1|} & \frac{1}{|x_3-x'_2|} & \frac{1}{|x_3-x'_3|} & \cdots & \frac{1}{|x_3-x'_{N-1}|} & \frac{1}{|x_3-x'_N|} \\ \vdots & & & \ddots & & \vdots \\ \frac{1}{|x_{N-1}-x'_1|} & \frac{1}{|x_{N-1}-x'_2|} & \frac{1}{|x_{N-1}-x'_3|} & \cdots & \frac{1}{|x_{N-1}-x'_{N-1}|} & \frac{1}{|x_{N-1}-x'_N|} \\ \frac{1}{|x_N-x'_1|} & \frac{1}{|x_N-x'_2|} & \frac{1}{|x_N-x'_3|} & \cdots & \frac{1}{|x_N-x'_{N-1}|} & \frac{1}{|x_N-x'_N|} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{N-1} \\ \alpha_N \end{bmatrix} = \begin{bmatrix} 4\pi\epsilon_0 V_0 \\ 4\pi\epsilon_0 V_0 \\ 4\pi\epsilon_0 V_0 \\ \vdots \\ 4\pi\epsilon_0 V_0 \\ 4\pi\epsilon_0 V_0 \end{bmatrix}. \quad (22)$$

Remember that all this expression represents is a series of voltage samples chosen along the wire such that the expansion coefficients drive the residual at each point to zero. This key step is the essence behind the method of moments. The only difficult part now is to choose an appropriate set of *test* locations at which to define x_m .

7 Test Functions

In principle, we are free to choose x_m along any locations where $V(x_m)$ is a known value. However in practice, it is desirable to pick values that are relatively spread out from each other in space. This helps to ensure a *well conditioned* matrix before attempting to calculate a solution. For example, we could theoretically choose all of our test locations around a tiny little neighborhood of the rod's midpoint using $x_m = L/2 + ma \times 10^{-9}$. Although this would still be mathematically possible to invert, it essentially places all of our test locations at the same point in space. The result of such a choice would be an *ill conditioned* matrix because each row would be nearly identical to all others. Numerical inversion therefore becomes impossible due to the finite precision of digital calculations.

For the case of our charge distribution problem, a seemingly natural choice is to define $x_m = m\Delta x/2$. In other words, we shall likewise place all x_m at the centers of each subdomain similar to the x'_n values. This allows us to write each matrix element using

$$\mathbf{A}_{mn} = \frac{1}{|x_m - x'_n|} , \quad (23)$$

where

$$\begin{aligned} x_m &= m\Delta x/2 , \\ x'_n &= n\Delta x/2 . \end{aligned}$$

Remember again that the x_m terms represent the *test* locations while x'_n represents the *source* locations.

The final step before attempting a matrix inversion is to handle the *self terms* in \mathbf{A} where $n = m$. Due to the overlap between the test and source locations, there naturally exist singularities along the diagonal elements of \mathbf{A} . Although this is generally not a problem for most applications, it is one that must be specifically dealt with for our charged rod. The ultimate problem arises from our assumption that the charge distribution consists of point sources located at discrete locations along the rod. In reality, a far better assumption is to treat ρ as a surface charge density that is spread out along a hollow tube with radius a . The voltage potential at the center of this tube is then found by integrating over the surface using

$$V(x_m) = \frac{1}{4\pi\epsilon_0} \int_{-\Delta x/2}^{+\Delta x/2} \int_0^{2\pi} \frac{\rho_n}{\sqrt{(x')^2 + a^2}} a d\phi' dx' . \quad (24)$$

Note that from our assumption of discrete charges at each sample, the expansion coefficient α_n represents the total net charge in a given segment. The surface charge density term ρ_n is therefore given by the total charge α_n divided by the surface area of the tube:

$$\rho_n = \frac{\alpha_n}{2\pi a \Delta x} . \quad (25)$$

Carrying out the integral then leads us to

$$V(x_m) = \frac{\alpha_m}{4\pi\epsilon_0 \Delta x} \ln \left| \frac{\Delta x + \sqrt{\Delta x^2 + 4a^2}}{\Delta x - \sqrt{\Delta x^2 + 4a^2}} \right| . \quad (26)$$

Finally, the diagonal matrix elements \mathbf{A}_{mm} for all of the self terms are found to be

$$\mathbf{A}_{mm} = \frac{1}{\Delta x} \ln \left| \frac{\Delta x + \sqrt{\Delta x^2 + 4a^2}}{\Delta x - \sqrt{\Delta x^2 + 4a^2}} \right| , \quad (27)$$

where the factor $4\pi\epsilon_0$ is assumed to have been moved to the right-hand side of the matrix equation. The vector of unknown expansion coefficients can now be found by solving for $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

8 Example

Consider a thin conductive wire with length $L = 1.0$ m and radius $a = L/40$. Figure 3 shows the resultant charge distribution $\hat{\rho}(x)$ after charging the rod to a uniform potential of $V_0 = 1.0$ V. The solution was achieved through the method of moments with delta functions as the basis for the charge distribution and $N = 40$ subdivisions. Notice how the charges tend to repel each other away from the center of the wire and pile up along the tips. Adding up the individual charges leads to a total charge of $q = 2.47$ pC.

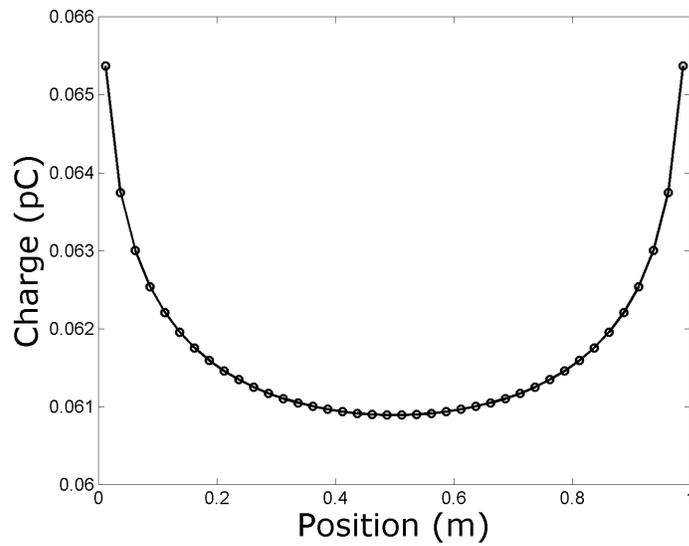


Figure 3: MoM solution to the charge distribution of a 1.0 m wire with radius $L/40$.

References

- [1] M. N. O. Sakidu, *Numerical Techniques in Electromagnetics with Matlab*, 3rd ed. Boca Raton, Fl: CRC Press, 2009.
- [2] L. L. Tsai and C. E. Smith, "Moment methods in electromagnetics for undergraduates," *IEEE Transactions on Education*, vol. E-21, no. 1, pp. 14–22, 1978.