Wave Propagation through a Layered Stack

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1 Introduction

Our physical system of interest is the stratified medium depicted in Figure 1. The model consists of M planar dielectric regions excited by a plane wave source from Region 1. Region 1 and Region M are infinite half-spaces, with the boundaries between each layer placed at the points d_1 through d_{M-1}. Each region is filled with a linear, isotropic medium, defined by a permittivity ϵ_i and permeability μ_i. For perfectly lossless materials, both ϵ_i and μ_i are purely real-valued. Imaginary components can also be added to account for loss. The complex index of refraction n_i for each region is then defined as

$$\tilde{n}_i = \frac{\sqrt{\mu_i \epsilon_i}}{\sqrt{\mu_0 \epsilon_0}} = n_i + j\kappa_i \ , \tag{1}$$

where ϵ_0 and μ_0 are the free-space permittivity and permeability. The parameter n_i is the real index of refraction while κ_i is the extinction coefficient.

Inside each region is a forward- and reverse-propagating plane wave with amplitude A_i and B_i, respectively. The incident field amplitude A_1 is a known value that is usually normalized to unit intensity. The excitation is also defined by a characteristic frequency of excitation f, but in practice it is common to use an equivalent free-space wavelength $\lambda_0 = c/f$, with c being the speed of light in a vacuum. We can likewise excite the system from an arbitrary angle of incidence θ with respect to the z-axis. Finally, we shall assume that there is no reverse-propagating wave in Region M, thereby setting $B_M = 0$. Using these given parameters, it is our goal to solve for all the field amplitudes throughout the system.

A comprehensive solution to this model can be found in many standard references, and the following derivation is based largely on the information found in Kong's book, *Electromagnetic Wave Theory* (2000). However, this discussion also includes many practical aspects of a stable numerical implementation that are difficult to find in the standard literature. This derivation also adheres to the convention of a time-dependent phasor notation with the form of $\text{Re}\{e^{kx}e^{-\omega t}\}$. This is consistent with the majority of the optics literature, but contrary to much of the RF and microwave literature. It also means that a positive value for the extinction coefficient κ implies a lossy material rather than a gain material. When calculating the time derivative of a phasor, we may further make the substitution $\partial/\partial t = -j\omega$.

Two possible polarization states exist for the incident field intensity. Under transverse-electric (TE) polarization, the incident electric field $\mathbf{E}$ is assumed to be y-polarized and A_1 is given as 1.0 V/m. For transverse-magnetic (TM) polarization, it is the magnetic field $\mathbf{H}$ that is assumed to be y-polarized and $A_1 = 1.0$ A/m. The derivation for each case is generally the same, so we shall first derive the TE case in detail and then quickly solve for the TM case.

2 TE Polarization

We begin with the case of TE polarization by expressing the forward-propagating wave in Region 1. Using phasor notation, this is written as

$$\mathbf{E}_1^+(x,z) = \hat{\mathbf{y}} A_1 e^{j(k_{1x}x + k_{1z}z)} \ . \tag{2}$$

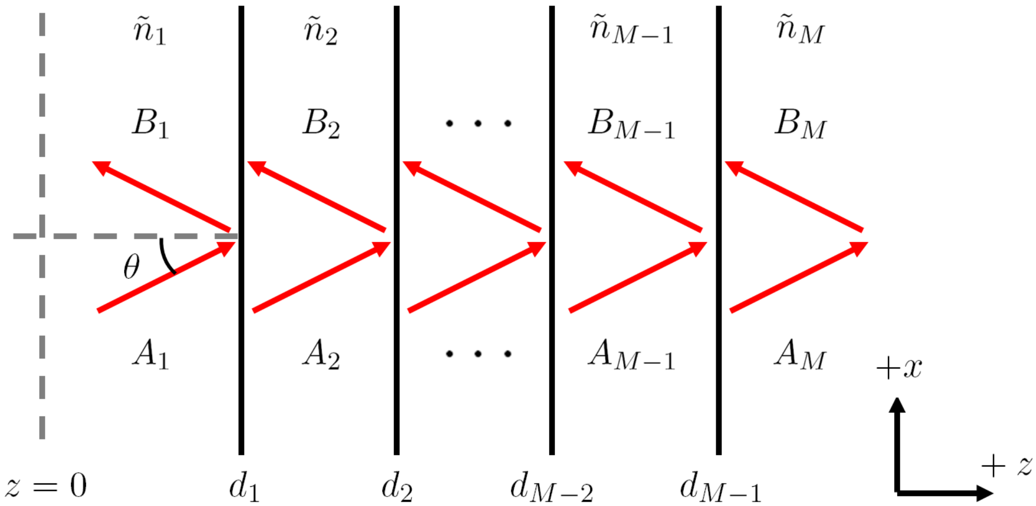

Figure 1: A stratified medium is excited by a plane wave from the left.

The parameters k_{1x} and k_{1z} denote the x- and z-components to the wavevector in Region 1. If we define the free-space wavenumber using $k_0 = 2\pi/\lambda_0$, then the wavevector components must satisfy

$$k_{1x}^2 + k_{1z}^2 = k_1^2 \ , \tag{3}$$

where $k_1 = k_0\tilde{n}_1$ is the wavenumber in Region 1. The wavenumber components are then related to the angle of incidence by

$$k_{1z} = k_0\tilde{n}_1 \cos\theta \ , \tag{4}$$
$$k_{1x} = k_0\tilde{n}_1 \sin\theta \ . \tag{5}$$

In a similar manner, we can now write out the expression for the reflected wave in Region 1 as

$$\mathbf{E}_1^-(x,z) = \hat{\mathbf{y}} B_1 e^{j(k_{1x}x - k_{1z}z)} \ . \tag{6}$$

Combining this with the forward propagating wave then gives us the total electric field for Region 1:

$$\begin{aligned}\mathbf{E}_1(x,z) &= \hat{\mathbf{y}} A_1 e^{j(k_{1x}x + k_{1z}z)} + \hat{\mathbf{y}} B_1 e^{j(k_{1x}x - k_{1z}z)} \ , \\ &= \hat{\mathbf{y}} \left(A_1 e^{+jk_{1z}z} + B_1 e^{-jk_{1z}z} \right) e^{jk_{1x}x} \ .\end{aligned} \tag{7}$$

Likewise, for any arbitrary region throughout the system, we can also write

$$\mathbf{E}_i(x,z) = \hat{\mathbf{y}} \left(A_i e^{+jk_{iz}z} + B_i e^{-jk_{iz}z} \right) e^{jk_{ix}x} \ . \tag{8}$$

The next step is to solve for the wavenumber components throughout each region. This is accomplished by enforcing the *phase-matching condition*, which simply states that all tangential wavevector components must be continuous across planar boundaries. The result of this condition is that $k_{ix} = k_{1x}$ for all layers, thereby allowing us to drop i subscript on k_x. Rewriting the total field in Region i then gives

$$\mathbf{E}_i(x,z) = \hat{\mathbf{y}} \left(A_i e^{+jk_{iz}z} + B_i e^{-jk_{iz}z} \right) e^{jk_x x} \ . \tag{9}$$

Finally, we can also solve for each k_{iz} component by applying the dispersion relation for each respective region:

$$k_{iz} = \sqrt{k_i^2 - k_x^2} = \sqrt{k_0^2 \tilde{n}_i^2 - k_x^2} \ . \tag{10}$$

This now gives us the complete solution to all wave vector components throughout the system.

Before continuing with the derivation, it is worth noting a practical issue of numerical stability. Although Equation (9) is a perfectly valid expression, it is also computationally unstable for materials with complex indices. The reason for this is because the phase on each plane wave is referenced with respect to $z = 0$. For a purely lossless system, this is normally not a problem and the computations are numerically stable. However, in the presence of gain or loss, there exists a magnitude disparity between the amplitude coefficients (A_i and B_i) and their complex exponential partners ($e^{+jk_{iz}}$ and $e^{-jk_{iz}}$). Numerically speaking, this can result in a multiplication between an extremely large number and an extremely small number. When the disparity stretches across enough orders of magnitude, accurate computation becomes impossible for a machine with finite digital precision.

To avoid the instabilities of numerical computation, it is helpful to reexpress Equation (9) by referencing the complex exponentials with respect to their plane of injection (i.e., the plane at the tail of each wavevector in Figure 1). This is accomplished by writing

$$\mathbf{E}_i(x,z) = \hat{\mathbf{y}} \left(A_i e^{+jk_{iz}(z-d_{i-1})} + B_i e^{-jk_{iz}(z-d_i)} \right) e^{jk_x x} \ . \tag{11}$$

An ambiguity arises under this convention because the points defined by d_0 and d_M are now arbitrary. We can solve this by declaring $d_0 = 0$, since this is a reasonably intuitive place for it to go. The choice of d_M is also irrelevant because the assumption $B_M = 0$ renders this point moot.

Now that we have settled on a proper notation, the next step is to solve for the field amplitudes throughout the system. This is accomplished by defining the magnetic field intensities in each region and then enforcing continuity of the tangential field components. We therefore begin with Faraday's law, which states that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = j\omega\mu\mathbf{H} \ . \tag{12}$$

Carrying out the curl operation within the ith region and solving for $\mathbf{H}_i$ then leads to

$$\begin{aligned} \mathbf{H}_i(x,z) = \ & \hat{\mathbf{z}} \frac{k_x}{\omega\mu_i} \left(+A_i e^{+jk_{iz}(z-d_{i-1})} + B_i e^{-jk_{iz}(z-d_i)} \right) e^{jk_x x} \\ & + \hat{\mathbf{x}} \frac{k_{iz}}{\omega\mu_i} \left(-A_i e^{+jk_{iz}(z-d_{i-1})} + B_i e^{-jk_{iz}(z-d_i)} \right) e^{jk_x x} \ . \end{aligned} \tag{13}$$

We are now ready to enforce continuity along the tangential field components at each planar boundary. For a given boundary at $z = d_i$, continuity on the electric field requires

$$a_i A_i + B_i = A_{i+1} + b_i B_{i+1} \ , \tag{14}$$

where, for notational compactness, the coefficients a_i and b_i are defined as

$$a_i = e^{+jk_{iz}(d_i - d_{i-1})} \ , \tag{15}$$

$$b_i = e^{-jk_{(i+1)z}(d_i - d_{i+1})} \ . \tag{16}$$

Note how a_i simply denotes the phase and amplitude change in a wave that exits the boundary at $z = d_{i-1}$ and propagates to d_i. The same is true for b_i, except for the reverse-propagating wave from d_{i+1} to d_i. If we likewise enforce continuity on the x-component to the magnetic field, we find

$$-\frac{k_{iz}}{\mu_i}\left(a_i A_i + B_i\right) = \frac{k_{(i+1)z}}{\mu_{i+1}}\left(-A_{i+1} + b_i B_{i+1}\right) \ . \tag{17}$$

Our goal now is to derive an expression for B_i/A_i. This process begins by first solving for $a_i A_i$ in Equation (14) and B_i in Equation (17):

$$a_i A_i = A_{i+1} + b_i B_{i+1} - B_i \ , \tag{18}$$

$$B_i = \frac{\mu_i}{\mu_{i+1}}\frac{k_{(i+1)z}}{k_{iz}}\left(-A_{i+1} + b_i B_{i+1}\right) + a_i A_i \ . \tag{19}$$

Cross-substitution between these two equations then leads to

$$A_i = \frac{1}{2a_i}\left(1 + \frac{\mu_i}{\mu_{i+1}}\frac{k_{(i+1)z}}{k_{iz}}\right)A_{i+1} + \frac{1}{2a_i}\left(1 - \frac{\mu_i}{\mu_{i+1}}\frac{k_{(i+1)z}}{k_{iz}}\right)B_{i+1}b_i \ , \tag{20}$$

$$B_i = \frac{1}{2}\left(1 - \frac{\mu_i}{\mu_{i+1}}\frac{k_{(i+1)z}}{k_{iz}}\right)A_{i+1} + \frac{1}{2}\left(1 + \frac{\mu_i}{\mu_{i+1}}\frac{k_{(i+1)z}}{k_{iz}}\right)B_{i+1}b_i \ . \tag{21}$$

Finally, divide these two expressions and simplify to arrive at

$$\frac{B_i}{A_i} = a_i\frac{\left(k_{iz}\mu_{i+1} - k_{(i+1)z}\mu_i\right) + b_i\left(k_{iz}\mu_{i+1} + k_{(i+1)z}\mu_i\right)\frac{B_{i+1}}{A_{i+1}}}{\left(k_{iz}\mu_{i+1} + k_{(i+1)z}\mu_i\right) + b_i\left(k_{iz}\mu_{i+1} - k_{(i+1)z}\mu_i\right)\frac{B_{i+1}}{A_{i+1}}} \ . \tag{22}$$

This is the essentially expression we are after, but we can still simplify it further by noting that the Fresnel reflecton coefficient for TE polarization between two media satisfies

$$\Gamma_{i(i+1)} = \frac{k_{iz}\mu_{i+1} - k_{(i+1)z}\mu_i}{k_{iz}\mu_{i+1} + k_{(i+1)z}\mu_i} \ . \tag{23}$$

Substitition finally leads us to a nice, compact result:

$$\boxed{\frac{B_i}{A_i} = a_i\frac{\Gamma_{i(i+1)} + b_i\left(\frac{B_{i+1}}{A_{i+1}}\right)}{1 + b_i\Gamma_{i(i+1)}\left(\frac{B_{i+1}}{A_{i+1}}\right)}} \ . \tag{24}$$

The utility of Equation (24) is that it provides a recursive relation between B_i/A_i and B_{i+1}/A_{i+1}. Since $B_M = 0$, the recursion terminates at $i = M$ under the condition $B_M/A_M = 0$. Equation (24) therefore provides a numerical algorithm for computing B_1/A_1. This algorithm is numerically stable for any combination of passive materials throughout the system, thus providing a reliable scheme for obtaining B_1. One only needs to exercise caution in the presence of gain materials within the system, which tend to numerically destabilize the calculations as a_i and b_i become excessively large.

Once B_1 is a known quantity, it can then be used to progressively solve for the rest of the wave amplitudes throughout the layers. To see how, consider the boundary at $z = d_1$. Assume also that A_i and B_i are now known values, which is certainly true for the $i = 1$ boundary. From basic theory of electromagnetic propagation between planar boundaries, we can now write

$$B_i = \Gamma_{i(i+1)}A_i a_i + \tau_{(i+1)i}B_{i+1}b_i \ , \tag{25}$$

where $\tau_{(i+1)i} = 1+\Gamma_{(i+1)i}$ is the Fresnel transmission coefficient (note the reversal of the indices due to the backward-propagation). In essense, this expression simply states that the reverse-propagating wave amplitude B_i at the ith boundary is determined by the reflected wave from Region i and the transmitted wave that propagates backward through Region $i+1$. We may therefore solve for B_{i+1} to arrive at

$$B_{i+1} = \frac{B_i - \Gamma_{i(i+1)} A_i a_i}{\tau_{(i+1)i} b_i} \ . \tag{26}$$

Finally, by a similar argument, we can also formulate the relation

$$A_{i+1} = \tau_{i(i+1)} A_i a_i + \Gamma_{(i+1)i} B_{i+1} b_i \ , \tag{27}$$

where again, the Fresnel coefficients are defined in a similar manner through a reversal in the subcript convention.

Together, Equations (26) and (27) provide an exact solution for the rest of the wave amplitudes throughout the system. Beginning with Region 1, the algorithm iterates from left to right solving for coefficients until reaching Region M. The solution then terminates with $B_M = 0$, giving with a complete, full-field solution for all wave amplitudes throughout the entire stratified medium. The advantage to this specific approach is again a matter of numerical stability. For regions with very high loss, the b_i term will take on a very small value and cause instability if Equation (26) is directly applied. Consequently, for very lossy regions, one may simply assume that $B_{i+1} = 0$ and then terminate the iterations after solving for A_{i+1}. In essence, all this says is that highly lossy regions will not have any reverse-propagating waves since all the energy is absorbed by the time the field propagates down and reflects back to the interface. A good cutoff point is perhaps $a_i < 10^{-9}$, which provides good accuracy for most desktop computers without introducing any significant numerical error.

Finally, it should be emphasized that this method is only accurate for an arbitrary combination of strictly passive layers. If the stratified medium contains a mixture of lossy and amplifying media, then numerical implementation becomes an intrinsically unstable problem for any method. The reason for this is the huge disparity of wave amplitudes that develop across several orders of magnitude within each layer. Because these must be numerically calculated with finite precision, accurate results may be impossible to achieve under extreme conditions. Fortunately, most real devices are almost exclusively comprised of passive layers, thereby guaranteeing accurate computations with this method.

3 TM Polarization

We now turn our attention to the case of TM polarization. By analogy with the previous section, the magnetic field in an arbitrary layer is expressed as

$$\mathbf{H}_i(x,z) = \hat{\mathbf{y}} \left(A_i e^{+jk_{iz}(z-d_{i-1})} + B_i e^{-jk_{iz}(z-d_i)} \right) e^{jk_x x} \ . \tag{28}$$

The electric field function is then determined from Ampere's law, given as

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = -j\omega\epsilon\mathbf{E} \ . \tag{29}$$

After solving for $\mathbf{E}_i$ in Region i, we then have

$$\begin{aligned} \mathbf{E}_i(x,z) = &-\hat{\mathbf{z}} \frac{k_x}{\omega\epsilon_i} \left(+A_i e^{+jk_{iz}(z-d_{i-1})} + B_i e^{-jk_{iz}(z-d_i)} \right) e^{jk_x x} \\ &+ \hat{\mathbf{x}} \frac{k_{iz}}{\omega\epsilon_i} \left(+A_i e^{+jk_{iz}(z-d_{i-1})} - B_i e^{-jk_{iz}(z-d_i)} \right) e^{jk_x x} \ . \end{aligned} \tag{30}$$

We are now ready to enforce continuity on the tangential field components at the $z = d_i$ boundary. Just like the TE case, the result is two coupled equations of the form

$$a_i A_i + B_i = A_{i+1} + b_i B_{i+1} \ , \tag{31}$$

$$\frac{k_{iz}}{\epsilon_i}\left(a_i A_i - B_i\right) = \frac{k_{(i+1)z}}{\epsilon_{i+1}}\left(A_{i+1} - b_i B_{i_1}\right) \ , \tag{32}$$

with the a_i and b_i terms defined by the same convention as before. Combining these two expressions and simplifying then leads to another recursion equation with the form

$$\frac{B_i}{A_i} = a_i \frac{\left(k_{iz}\epsilon_{i+1} - k_{(i+1)z}\epsilon_i\right) + b_i\left(k_{iz}\epsilon_{i+1} + k_{(i+1)z}\epsilon_i\right)\frac{B_{i+1}}{A_{i+1}}}{\left(k_{iz}\epsilon_{i+1} + k_{(i+1)z}\epsilon_i\right) + b_i\left(k_{iz}\tilde{n}_{i+1}^2 - k_{(i+1)z}\epsilon_i\right)\frac{B_{i+1}}{A_{i+1}}} \ . \tag{33}$$

Once again, this can be simplified by defining the TM Fresnel coefficients

$$\Gamma'_{i(i+1)} = \frac{k_{iz}\epsilon_{i+1} - k_{(i+1)z}\epsilon_i}{k_{iz}\epsilon_{i+1} + k_{(i+1)z}\epsilon_i} \ , \tag{34}$$

$$\tau'_{(i+1)i} = 1 + \Gamma'_{(i+1)i} \ . \tag{35}$$

The recursion relation then takes on the familiar form

$$\boxed{\frac{B_i}{A_i} = a_i \frac{\Gamma'_{i(i+1)} + b_i\left(\frac{B_{i+1}}{A_{i+1}}\right)}{1 + b_i\Gamma'_{i(i+1)}\left(\frac{B_{i+1}}{A_{i+1}}\right)}} \ . \tag{36}$$

With B_1 a known value, we can again solve for the rest of the wave amplitudes using Equations (26) and (27). The only change is in the Fresnel coefficients, which now take on their primed counterparts. We therefore have a complete, full-field solution to the problem of plane wave excitation of a planar, stratified medium. The solution is numerically stable under any combination of passive materials, incident angle, and wave polarization.